r = radial position
r_1, r_2 = inner and outer radii of the annulus
r_H = $(r_2 - r_1)/2$ = hydraulic radius
Re = Reynolds number
u = local velocity in the x direction
u_m = maximum velocity in the entrance region
u_{mf} = maximum velocity in the fully developed flow
$\langle u \rangle$ = average velocity
Ws = $(\sigma/K)(\langle u \rangle/r_H)^{s-n}$, Weissenberg number
x = axial distance measured from the contraction plane
χ = $x/r_H Re$, dimensionless axial distance
ξ = Ws/Re = elasticity number = elastic force/inertial force
$\dot{\gamma}$ = $-du/dr$, shear rate in steady shear flow
τ = shear stress

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Periodic Flow in a Curved Tube

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Steady Poiseuille flow in a rigid curved tube and its effect on associated transport processes have been studied extensively since Dean (1927) first predicted the characteristic twin vortex secondary motion (Figure 1a) associated with this flow (see, for example, Tarbell and Samuels 1973, Kalb and Seader 1974, and Smith 1976). However, periodic Poiseuille flow in a rigid curved tube was first analyzed only recently by Lyne (1970) who considered the problem of pure sinusoidal flow, with zero mean. Lyne constructed an asymptotic expansion solution of the Navier-Stokes equations, valid for high frequencies, $\alpha >> 1$, and obtained a striking result. For sufficiently high frequencies ($\alpha > 12.9$), the twin vortex motion which characterizes steady flow transforms to a qualitatively new four-vortex motion (Figure 1b). Inward centrifuging near the center of the tube was quite unexpected. Independent theoretical studies of this same problem by Zalosh and Nelson (1973) and Chandran (1974), employing quite distinct solution techniques, predicted the same new four-vortex motion. These theoretical predictions of the new phenomenon were subsequently verified experimentally by Bertelsen (1975).

In this note, we report numerical results on the problem of periodic Poiseuille flow (*with non-zero mean*) in a rigid curved tube. The only previous work on this problem was by Smith (1975), who used a variety of asymptotic expansions to obtain approximate solutions in the high ($\alpha >> 1$) and low ($\alpha << 1$) frequency regimes. We believe that our numerical results for intermediate frequencies reveal a significant new phenomenon—*resonance between the axial flow and the secondary flow.* This means that the secondary flow has a natural frequency, approximately equal to the circulation time of the secondary flow at the time-averaged flow rate, which is excited by the oscillating axial flow over a narrow range of frequencies. The character of the resonating flow may be quite unusual and not anticipated on the basis of low frequency (quasi-steady) or high frequency (relaxed steady) information. It must be emphasized that the four-vortex secondary flow discovered by

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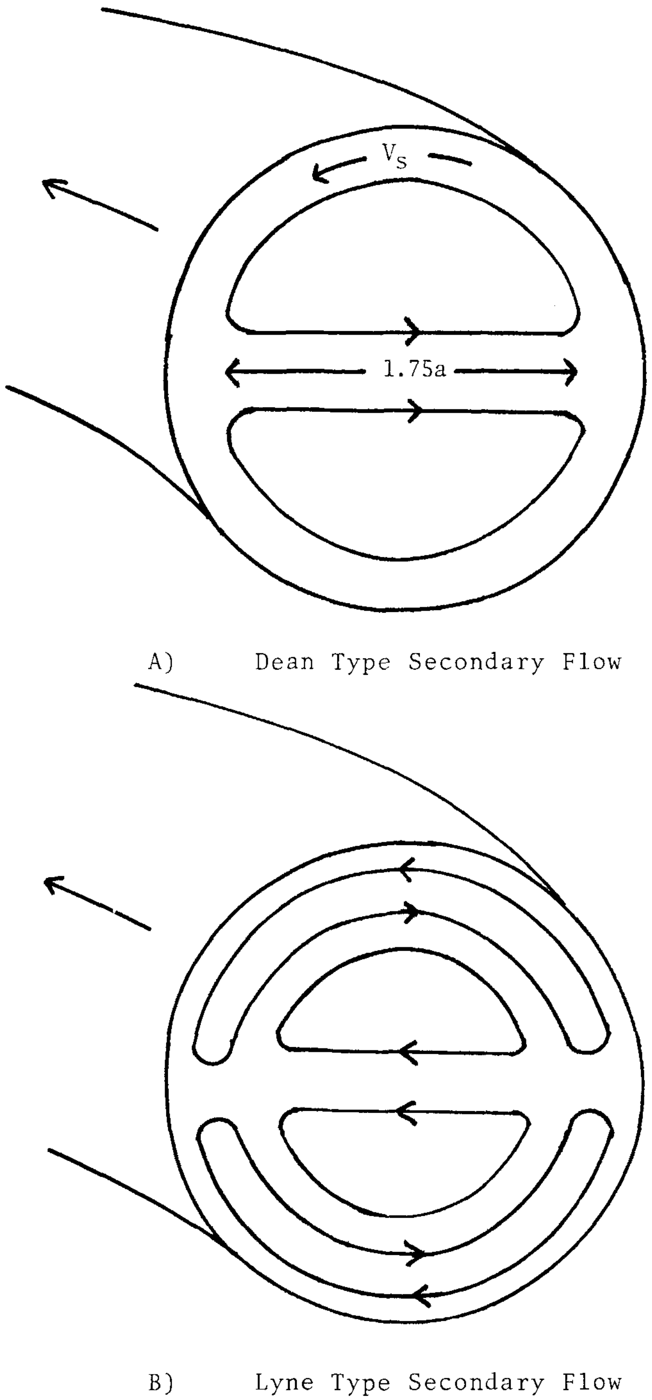

Figure 1. Curved tube secondary flows.

Lyne (1970) is not a manifestation of resonance, since a pure sinusoidal flow contains no underlying steady motion and thus no characteristic circulation time (natural frequency).

NUMERICAL METHODS

Solutions of the Navier-Stokes and continuity equations for fully developed laminar flow in a rigid curved tube with an imposed axial pressure gradient having the dimensionless form, $K(1 + k \sin \omega t)$, were obtained by a modified "alternating direction implicit" algorithm. The equations, the grid work, and the *basic algorithm* were described by Tarbell and Samuels (1973) in connection with the steady flow ($k = 0$) version of the problem. In the periodic flow algorithm, the time period $[0, 2\pi/\omega]$ was divided into N equal increments, and the *basic algorithm* was employed to integrate all fields over one period. With iteration to convergence of the stream function-vorticity equation at every time step.) During this integration, field values from the "old" period (which unfortunately had to be stored in their entirety) were incorporated in explicit calculations whenever "new" period values were not available. The "new" period fields were compared to the "old" period fields, and if adequate convergence was not apparent, additional iterations were pursued. Convergence was considered adequate when the time-averaged ($\sim$) and peripherally averaged ($-$) friction factor, $\tilde{\bar{f}}(\lambda, K, k, \alpha)$, had converged to within 10^{-4} on a relative basis.

The algorithm was capable of accurately reproducing the steady flow results of Tarbell and Samuels (1973) for the few low D_n cases that were tested. It was also able to accurately generate the theoretical results of Womersley (1955) and Uchida (1956) for fully developed, periodic flow in a straight, rigid tube when the aspect ratio (λ) was set at a large value. Thus, we have confidence in the accuracy and consistency of the results to be discussed. Unfortunately, our ability to obtain results with reasonable computation time and storage requirements was limited by numerical stability considerations. Our experience with the algorithm indicates that the following rough stability criterion must be satisfied:

$$\alpha^2 N > 200\,\pi \qquad (1)$$

Since the storage limits of the IBM 370-3033 (560K bytes) were approached for $N \sim 20$, we were unable to obtain solutions for $\alpha < 5$.

Results obtained by the above method are supplemented by values of the time- and peripherally averaged friction factor in the low frequency ($\alpha \to 0$) limit. These values are easily obtained because the periodic flow approaches a quasi-steady state as $\alpha \to 0$. This limiting low frequency friction factor

$$\tilde{\bar{f}}_o(\lambda, K, k) \equiv \lim_{\alpha \to 0} \tilde{\bar{f}}(\lambda, K, k, \alpha) \qquad (2)$$

is calculated as follows:

$$\tilde{\bar{f}}_o(\lambda, K, k) = \frac{K}{\lambda <\tilde{w}_s>^2(\lambda, K, k)} \qquad (3)$$

where

$$<\tilde{w}_s>(\lambda, K, k) = \frac{1}{2\pi}\int_0^{2\pi} <w_s>(\lambda, K(1 + k \sin \tau))\,d\tau \qquad (4)$$

and $<w_s>(\lambda, K)$ is the steady flow, area averaged, axial velocity which depends only on the aspect ratio (λ) and the steady pressure gradient (K). The $<w_s>(\lambda, K)$ values of Tarbell and Samuels (1973) and a simple Simpson's rule technique were employed to evaluate the integral in Equation (4).

RESULTS

Figure 2 contains results obtained by the methods of the last section. The ordinate in Figure 2 is the ratio of the time- and peripherally averaged friction factor under periodic pressure gradient, $\tilde{\bar{f}}(K, k, \alpha)$, to the peripherally averaged friction factor under steady pressure gradient, $\bar{f}(K)$, while the abscissa is a dimensionless frequency variable, α. The curves are not complete, and the dashed portions represent regions where we have not been able to obtain solutions for the reasons discussed previously. However, available results are sufficient to demonstrate that there are maxima in the curves, although the dashed portions may represent exaggerations of the actual phenomenon (resonance).

Evidence to support a hypothesis that the maxima in the friction factor versus frequency curves are a manifestation of resonance between the axial flow and the secondary flow comes from two sources. First, analytical solutions of the Navier-Stokes equations for fully developed periodic flow in rigid straight tubes (Womersley 1955, Uchida 1956) predict a time-averaged friction factor which is independent of frequency and amplitude (see Figure 2). This is significant because the straight tube flow has no secondary motion—no natural frequency. Second, estimates (derived later on) of the characteristic circulation time of the secondary motion in a curved tube flow under steady (time-averaged) pressure gradient (K) predict the resonant frequencies (α_{res}) shown in Figure 2. Clearly the maxima in the friction factor versus frequency curves must occur at frequencies near the predicted α_{res}.

An estimate of α_{res} is developed from the definition of the resonant frequency

$$\omega_{res} \equiv 2\pi \frac{V_s}{L_s} \tag{5}$$

L_s is a characteristic path length and V_s is a characteristic path velocity for the steady secondary flow (see Figure 1a). The following estimates are based on the work of Dravid et al. (1971):

$$L_s = 4.5a \tag{6a}$$

$$V_s = \frac{\nu}{a} (0.9656\, \tilde{D}_n^{1/2} + 1.65) \tag{6b}$$

where $\tilde{D}_n$ is the time-averaged Dean number. Equations (5) and (6a, b) lead to the final estimate

$$\alpha_{res} = 1.18(0.9656\, \tilde{D}_n^{1/2} + 1.65)^{1/2} \tag{7}$$

A few final remarks concerning Figure 2 may be worthwhile. The damping of flow oscillations with increasing frequency is apparent, as the curves for various amplitudes (k's) at the same mean pressure gradient (K) converge at high frequency. The instantaneous secondary flow for all computed cases was of the Dean type. No exotic Lyne type secondary flows were observed.

IMPLICATIONS

Equation (7) indicates that α_{res} is a weak function of time-averaged flow rate. At $\tilde{D}_n = 50$, $\alpha_{res} = 3.43$, while at $\tilde{D}_n = 1000$, $\alpha_{res} = 6.69$. In the adult human cardiovascular system, α ranges from ~10 in large arteries down to <<1 in capillaries. Thus, we may expect that resonance occurs in large curved arteries like the aorta. It is interesting to note that the aorta is one of the focal regions for proliferation of artherosclerosis, a disease whose pathogenesis is thought to be linked with local fluid mechanics and radial mass transport processes (see the representative theories of Fry 1969, Caro et al. 1971).

Engineering applications of resonance are suggested by the results of Figure 2. Resonance of the fluid mechanics

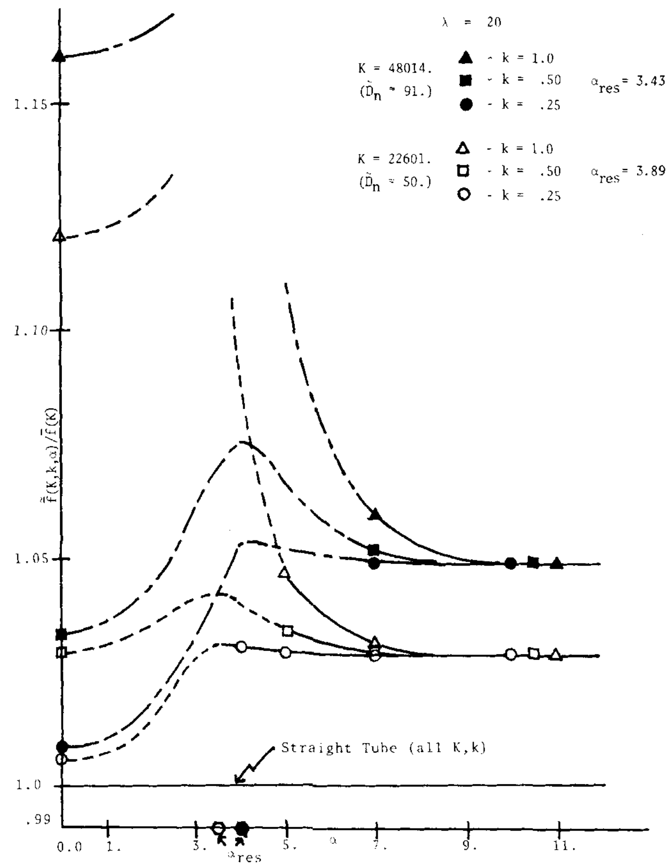

Figure 2. Friction factor for periodic flow in a curved tube.

may induce resonance of the radial heat and mass transfer rates resulting in enhanced Nusselt numbers for frequencies near the resonant frequency. If the enhancement is significant, it might be exploited in the design of compact heat and mass transfer equipment, where the penalty for additional power requirements is not prohibitive.

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NOTATION

a = tube radius
D_n = Dean number $\equiv Re/\sqrt{\lambda}$
$f(f_o)$ = friction factor ($\alpha \to 0$)
K = steady axial pressure gradient (dimensionless)
k = amplitude of oscillatory pressure gradient (dimensionless)
L_s = characteristic path length for the steady secondary flow
N = number of time increments in one period
R = radius of curvature
Re = Reynolds number
t = time
V_s = characteristic path velocity for the steady secondary flow
w_s = steady flow axial velocity (dimensionless)

Greek Letters

α = frequency parameter $\equiv a\sqrt{\omega/\nu}$
α_{res} = resonant value of α
ν = kinematic viscosity
λ = aspect ratio $\equiv R/a$

τ = dimensionless time $\equiv \omega t$
ω = oscillation frequency
ω_{res} = resonant value of ω

Symbols

$< >$ = cross-sectional average
$-$ = peripheral average
$\sim$ = time average

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Error in the Propagation of Error Formula

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To predict the mean and variance of a dependent variable from the ensemble (or sample) means of the independent variables in a process model, engineers frequently use the "propagation of error" relation, shown in Equation (2) below. For linear models, no difficulty arises in using this relationship. However, we wish to point out that it can yield quite misleading results if the dependent variable in the process model is a nonlinear function of the independent variables.

To avoid complexity in what follows, we assume that the independent variables in the process model are also statistically independent so that crossproduct terms in the propagation of error relation can be deleted.

Let us express the process model as a function of several random variables $X_1, X_2, \ldots, X_n$ with $Y = f(\mathbf{X})$ representing the dependent variable. The usual procedure is to linearize $f(\mathbf{X})$ by a Taylor series

$$Y \simeq f(\mathbf{x}_0) + \sum_{i=1}^{n} \frac{\delta f(\mathbf{x}_0)}{\delta x_i}(X_i - x_{i0}) \qquad (1)$$

and thereafter compute the Var $\{Y\} \equiv \sigma^2{}_Y$ by

$$\text{Var}\{Y\} \simeq \sum_{i=1}^{n} \left[\frac{\delta f(\mathbf{x}_0)}{\delta x_i}\right]^2 \text{Var}\{X_i\} \qquad (2)$$

where x_{i0} is a deterministic reference value, such as the value of the ensemble or sample mean of X_i.

If the ensemble means μ_{X_i}, and the ensemble variances $\sigma^2 x_i \equiv \text{Var}\{X_i\}$ are substituted into (1) and (2) to approximate μ_Y and $\sigma^2{}_Y$, we can show by a simple example how distorted the approximations can be. As an example, we calculate the equilibrium constant from data for the standard free energies at a temperature T

$$K = \exp\left(-\frac{\Delta F^\circ}{RT}\right) \qquad (3)$$

Let us assume the random variable ΔF° can be represented by a normal distribution with a known ensemble mean $\mu_{\Delta F^\circ}$ and variance $\sigma^2{}_{\Delta F^\circ}$. Then

$$\mu_K \equiv \mathcal{E}\{K\} = \mathcal{E}\left\{\exp\left(-\frac{\Delta F^\circ}{RT}\right)\right\}$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{\Delta F^\circ}{RT}\right) p(\Delta F^\circ) d(\Delta F^\circ)$$

where

$$p(\Delta F^\circ) = \frac{1}{\sqrt{2\pi}\,\sigma_{\Delta F^\circ}} \exp\left[-\tfrac{1}{2}\left(\frac{\Delta F^\circ - \mu_{\Delta F^\circ}}{\sigma_{\Delta F^\circ}}\right)^2\right]$$

After integration we find

$$\mu_K = \exp\left[-\frac{\mu_{\Delta F^\circ}}{RT} + \tfrac{1}{2}\left(\frac{\sigma_{\Delta F^\circ}}{RT}\right)^2\right] \qquad (4)$$

Similarly,

$$\sigma_K{}^2 \equiv \text{Var}\{K\} = \int_{-\infty}^{\infty} (K - \mu_K)^2 p(K) dK$$